

# Methods for Estimating Capacities of Gaussian Quantum Channels

Oleg V. Pilyavets,<sup>1,2,\*</sup> Cosmo Lupo,<sup>1,†</sup> and Stefano Mancini<sup>1,‡</sup>

<sup>1</sup>*Dipartimento di Fisica, Università di Camerino, I-62032 Camerino, Italy*

<sup>2</sup>*P. N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia*

We present a perturbative approach to the problem of estimating capacities of Gaussian quantum channels. It relies on the expansion of the von Neumann entropy of Gaussian states as a function of the symplectic eigenvalues of the quadratures covariance matrices. We apply this method to the classical capacity of a lossy bosonic channel for both the memory and memoryless cases.

Keywords: Quantum information, Gaussian quantum channels, Classical capacity of quantum channels.

## I. INTRODUCTION

Quantum channels are every means that convey quantum systems on whose states information is encoded. Formally it is a quantum map from input to output states [1]. The maximum rate at which information can be reliably transmitted through a quantum channel defines its capacity. Actually one can define several capacities depending on the kind of information transmitted (classical or quantum) and on the additional resources used in transmission [2].

Evaluation of quantum channels capacities is one of the most important and difficult problems of quantum information theory. Gaussian channels, which maps input Gaussian states into output Gaussian states, are among the simplest models allowing capacities investigation [3]. They are also relevant for experimental implementations [4].

A paradigmatic example of quantum Gaussian channel is the lossy bosonic channel where states lose energy en route from the sender to the receiver. The term bosonic arises because each input (respectively output) is represented by a bosonic field mode. In turn, the effect of losses is usually modeled by letting each input mode to interact with an environment mode through a rotation (beam splitter) transform whose angle (transmissivity) determines the loss rate [4].

The classical capacity and the classical assisted capacity for such a channel were evaluated in Refs. [5, 6] by assuming each environment mode in the vacuum state. In such a case Gaussian states turn out to be the optimal inputs. Then it has been proved that Gaussian inputs suffices to determine the quantum capacity [7]. More recently, it has been proven that they are optimal to reach the classical capacity when environment modes are in thermal states (including the vacuum) [8]. When more general states of environment are taken into account, even non factorized ones which give rise to memory effect [9], the evaluation of capacities becomes much more demanding.

Here we develop a perturbative approach by expanding the von Neumann entropy of Gaussian quantum state as a function of the symplectic eigenvalues of its quadratures covariance matrix. The method turns out to be useful for characterizing capacities of Gaussian quantum channels. For the sake of presentation, we apply it to the classical capacity of a lossy bosonic channel. We then obtain lower bounds for the classical capacity, generalizing the results of Refs. [3, 6].

The paper is organized as follows. In Sec. II the main features of the lossy bosonic channel model are introduced. In Sec. III the general algorithm to find lower bounds on the classical capacity is presented. In Sec. IV the lower bound for the memoryless lossy bosonic channel is given. In Sec. V the lower bound for the classical capacity is found for a memory lossy bosonic channel. Sec. VI is for conclusions.

## II. THE LOSSY BOSONIC CHANNEL

Let us start from the consideration that in the evaluation of capacities of a Gaussian quantum channel one deals with the von Neumann entropy of Gaussian states, involving the function [3]

$$g\left(v - \frac{1}{2}\right) := \left(v + \frac{1}{2}\right) \log_2 \left(v + \frac{1}{2}\right) - \left(v - \frac{1}{2}\right) \log_2 \left(v - \frac{1}{2}\right), \quad (1)$$

where  $v$  is a symplectic eigenvalue of the covariance matrix characterizing a Gaussian state. Despite the absence of a small parameter for the expansion of the function  $g$ , we proceed as follows. The function  $g(v - 1/2)$  is not analytic in neighborhood of  $1/2$  and infinity, however after subtracting the logarithm part it becomes analytic on the region  $v \geq 1/2$ . Hence, we may write (see also [10])

$$g\left(v - \frac{1}{2}\right) = \log_2 v + \frac{1}{\ln 2} \left[ 1 - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(2v)^{-2j}}{j(2j+1)} \right]. \quad (2)$$

Then, to the zeroth-order approximation we have

$$g\left(v - \frac{1}{2}\right) \approx \log_2 v + \frac{1}{\ln 2}, \quad (3)$$

\*Electronic address: pilyavets@gmail.com

†Electronic address: cosmo.lupo@unicam.it

‡Electronic address: stefano.mancini@unicam.it

where we have neglected terms of the order  $O(1/v^2)$ . Allowing perturbation of logarithm by the first terms in the series (2) we can also construct next-order approximations.

Below we restrict our attention to the classical capacity of a lossy bosonic channel. Since we shall consider Gaussian states, the action of this channel can be regarded as a linear map on the covariance matrices characterizing such states. We label covariance matrices acting on canonical quadratures  $(q_1, \dots, q_n, p_1, \dots, p_n)^\top$  by  $V_{\text{in}}$ ,  $V_{\text{env}}$  and  $V_{\text{out}}$  for input, environment and output states of the channel respectively.  $V_{\text{cl}}/2$  denotes the covariance matrix for the distribution of coherent amplitudes used to encode classical symbols.  $\bar{V}_{\text{out}}$  is the state of channel output averaged over encoding distribution (with covariance  $V_{\text{cl}}/2$ ). Also, we will refer to the eigenvalues of matrices  $V_{\text{in}}$ ,  $V_{\text{env}}$ ,  $V_{\text{out}}$ ,  $V_{\text{cl}}$  and  $\bar{V}_{\text{out}}$  as  $i_{uk}$ ,  $e_{uk}$ ,  $o_{uk}$ ,  $c_{uk}$  and  $a_{uk}$  respectively. Since  $n \in \mathbb{N}$  represents the number of bosonic modes (channel uses), the dimension of all above matrices is  $2n \times 2n$  and  $k = 1, \dots, n$ , while  $u$  denotes either quadrature  $q$  or  $p$  (if  $u = q$ , then  $u_\star = p$ , and vice versa). In particular the lossy bosonic channel acting as a rotation (beam splitter) transform on the canonical quadratures, gives rise to the following relations among covariance matrices [11]:

$$V_{\text{out}} = \eta V_{\text{in}} + (1 - \eta) V_{\text{env}}, \quad (4)$$

$$\bar{V}_{\text{out}} = \eta (V_{\text{in}} + V_{\text{cl}}) + (1 - \eta) V_{\text{env}}, \quad (5)$$

where  $\eta$  represents the transmissivity of the channel. As usual (see e.g. [9]) we shall constrain the input energy by

$$\frac{1}{2n} \text{Tr}(V_{\text{in}} + V_{\text{cl}}) = N + \frac{1}{2}, \quad (6)$$

with  $N$  the average number of input photons per mode (channel use).

An upper bound on the classical capacity is provided by the maximum of the Holevo information regularized in the limit of infinite channel uses

$$\bar{C} := \lim_{n \rightarrow \infty} \bar{C}_n; \quad \bar{C}_n := \frac{1}{n} \max \chi_n \quad (7)$$

where the maximum is taken over *all* input states satisfying the energy constraint (6) and  $\chi_n$  is the Holevo- $\chi$  quantity [12, 13]. By coding theorems [12, 13], the quantity  $\bar{C}$  turns out to exactly be the capacity for memoryless and forgetful channels [14, 15].

For any  $n$ , one can look at  $n$  uses of the channel described by Eqs. (4) and (5) as a single channel acting on  $n$  modes. Then, a lower bound on the classical capacity can be computed by maximizing its *one-shot* capacity over Gaussian input states [16]

$$\underline{C}_n := \frac{1}{n} \max_G \chi_n. \quad (8)$$

The Holevo- $\chi$  quantity for the set  $G$  of Gaussian states reads [3]

$$\chi_n = \sum_{k=1}^n \left[ g\left(\bar{\nu}_k - \frac{1}{2}\right) - g\left(\nu_k - \frac{1}{2}\right) \right], \quad (9)$$

where  $\bar{\nu}_k$  and  $\nu_k$  are the symplectic eigenvalues of  $\bar{V}_{\text{out}}$  and  $V_{\text{out}}$  respectively. Taking the limit over  $n$ , we can as well define the quantity

$$\underline{C} := \lim_{n \rightarrow \infty} \underline{C}_n. \quad (10)$$

We first notice that by using (3) we have, at lowest order,

$$\chi_n^{(0)} = \sum_{k=1}^n \log_2 \frac{\bar{\nu}_k}{\nu_k}. \quad (11)$$

Then, we restrict our environment model to matrices of the form

$$V_{\text{env}} = \begin{pmatrix} V_{qq} & 0 \\ 0 & V_{pp} \end{pmatrix} \quad (12)$$

with commuting blocks  $V_{qq}$  and  $V_{pp}$ . In this case we conjecture that the maximum of  $\chi$ -quantity (9) is achieved with matrices  $V_{\text{in}}$  and  $V_{\text{cl}}$  of the same form as (12), *i.e.* with null off-diagonal blocks. Furthermore, all diagonal blocks of all matrices are mutually commuting. This conjecture is supported by numerical investigations relying on environment models of the form (12). As a consequence, we get the following expressions for symplectic eigenvalues

$$\nu_k = \sqrt{o_{qk} o_{pk}}, \quad \bar{\nu}_k = \sqrt{a_{qk} a_{pk}}. \quad (13)$$

Notice that both energy constraint (6) and symplectic spectrum (13) are preserved under orthogonal transformations. Thus, without affecting the final result, we can consider all the involved matrices to be diagonal (see also the discussion in the appendix of [16]).

Hence, the problem we are going to consider is the maximization (8) of the Holevo function (9) over Gaussian inputs under the constraints

$$\frac{1}{2n} \sum_{k=1}^n [i_{uk} + i_{u_\star k} + c_{uk} + c_{u_\star k}] = N + \frac{1}{2}, \quad (14)$$

$$i_{uk} > 0, \quad (15)$$

$$c_{uk} \geq 0, \quad (16)$$

$$i_{uk} i_{u_\star k} \geq \frac{1}{4}, \quad (17)$$

with  $\nu_k$ ,  $\bar{\nu}_k$  given by Eqs. (13), where

$$o_{uk} = \eta i_{uk} + (1 - \eta) e_{uk},$$

$$a_{uk} = \eta (i_{uk} + c_{uk}) + (1 - \eta) e_{uk}.$$

Here, the condition (14) is the input energy constraint (6), while inequalities (15) and (17) express the Heisenberg uncertainty relation.

### III. THE KARUSH-KUHN-TUCKER EQUATIONS

The above maximization problem with constraints can be solved by considering eigenvalues of our matrices as

variables for Karush-Kuhn-Tucker (KKT) conditions in analogy with the same problem for classical Gaussian channels with memory [17]. To simplify the calculations we will apply a technique similar to the simplex method in optimization theory. Assuming monotonic dependence on  $N$  of the eigenvalues  $\{c_{uk}\}$  maximizing  $\chi$  (which is supported by numerical investigation), we do not specify their sign in KKT-equations. At first, we will consider the formal solution of KKT-equations assuming negative eigenvalues as result, then we will specify how to eliminate them.

**Definition 1.** *Let us consider the eigenvalues  $c_{uk}$ ,  $u \in \{q, p\}$  maximizing  $\chi$  as monotonically dependent functions of  $N \in [0, \infty)$ , then we will classify the  $k$ -th mode as belonging to the:*

- first stage *iff*  $c_{qk}, c_{pk} = 0$ ;
- second stage *iff*  $c_{uk} = 0$ ,  $c_{u^*,k} > 0$ ;
- third stage *iff*  $c_{qk}, c_{pk} > 0$ .

By increasing  $N$  starting from zero we allow each mode to pass all stages in sequence from the first one up to the third one.

The Lagrange function for  $n$ -modes channel reads

$$L = -\frac{\chi_n}{n} + \sum_{k=1}^n \mu_k \left( \frac{1}{4} - i_{qk} i_{pk} \right) + \lambda \left( \frac{1}{n} \sum_{k=1}^n [i_{qk} + i_{pk} + c_{qk} + c_{pk}] - 2N - 1 \right).$$

Solving the KKT-conditions we find that the input state  $V_{\text{in}}$  maximizing  $\chi$  must be pure. When all modes belong to the third stage, the eigenvalues of  $\bar{V}_{\text{out}}$  are all equal and the solution reads

$$c_{uk} = N + \frac{1}{2} - \frac{1}{2} \sqrt{\frac{e_{uk}}{e_{u^*,k}}} + \frac{1-\eta}{\eta} \left( \frac{\text{Tr} V_{\text{env}}}{2n} - e_{uk} \right), \quad (18)$$

$$i_{uk} = \frac{1}{2} \sqrt{\frac{e_{uk}}{e_{u^*,k}}}, \quad (19)$$

giving

$$\underline{C}_n = g[\eta N + (1-\eta)M_{\text{env}}] - \frac{1}{n} \sum_{k=1}^n g[(1-\eta)(\sqrt{e_{qk}e_{pk}} - 1/2)], \quad (20)$$

where

$$M_{\text{env}} := \frac{\text{Tr} V_{\text{env}}}{2n} - \frac{1}{2} \quad (21)$$

is the average number of photons in the environment. The analytical lower bound in (20) generalizes the expression presented in [16].

If we consider the lower bound (20) as a function on the set of memory models characterized by different  $V_{\text{env}}$  each of them with the same  $M_{\text{env}}$ , we argue from Eq. (20) that the maximum is achieved when the state  $V_{\text{env}}$  is pure.

We now move to the general case. At first, we assume that the correct distribution of modes over stages is already found. Then, suppose to know that the eigenvalue  $a_{u^*,h}$  (for the mode  $h$ ) belongs to the second stage and  $c_{uh} = 0$ , it follows the transcendent equation for  $i_{uh}$

$$\left( \frac{1}{o_{uh}} - \frac{1}{a_{u^*,h}} \right) \bar{\nu}_h \frac{\partial g(\bar{\nu}_h - 1/2)}{\partial \bar{\nu}_h} = \left( \frac{1}{o_{uh}} - \frac{i_{u^*,h}}{i_{uh} o_{u^*,h}} \right) \nu_h \frac{\partial g(\nu_h - 1/2)}{\partial \nu_h}, \quad (22)$$

where  $c_{u^*,h} = (a_{u^*,h} - (1-\eta)e_{u^*,h})/\eta - 1/(4i_{uh})$  and  $i_{u^*,h} = 1/(4i_{uh})$ . This equation has real roots if both  $\bar{\nu}_h, \nu_h > 1/2$ , giving

$$i_{uh} > \max \left\{ \frac{1}{\eta} \left( \frac{1}{4a_{u^*,h}} - (1-\eta)e_{uh} \right), 0 \right\}. \quad (23)$$

In turn, the requirement  $\bar{\nu}_h > \nu_h$  leads to the inequality

$$i_{uh} > \frac{\eta}{4(a_{u^*,h} - (1-\eta)e_{u^*,h})}. \quad (24)$$

Depending on the value of  $a_{u^*,h}$ , equation (22) can admit one root satisfying (24) or none.

Equation (22) can be formally written as the dependence  $a_{u^*,h} = f_h(i_{uh})$ . Let  $\lambda$  be the only parameter linking the KKT-equations of different modes, we define a new variable  $x$

$$x := a_{qm} = a_{pm} = a_{ql} = a_{pl} = \dots = a_{qh} = f_h(i_{ph}) = a_{pt} = f_t(i_{qt}), \quad (25)$$

getting a chain of equalities linking *all* modes of the second and third stages. Here modes  $m$  and  $l$  belong to the third stage, while modes  $h$  and  $t$  to the second stage ( $c_{ph} = c_{qt} = 0$ ). Modes of the first stage are not included in (25) and all give  $V_{\text{in}}$ -eigenvalues equal to  $1/2$ . If some mode belongs to the third stage, its  $V_{\text{in}}$ -eigenvalues can be found from the relation (19).

Taking into account stages discrimination, equation (5) can be rewritten as

$$\eta \text{Tr}_{\{2,3|c_{uk} \neq 0\}} \bar{V}_{\text{in}} + (1-\eta) \text{Tr}_{\{2,3|c_{uk} \neq 0\}} V_{\text{env}} = [2n_3 + n_2]x, \quad (26)$$

where  $\bar{V}_{\text{in}} := V_{\text{in}} + V_{\text{cl}}$ . Furthermore,  $n_j$  is the number of modes belonging to  $j$ -th stage ( $j = 1, 2, 3$ ;  $n = n_1 + n_2 + n_3$ ) and  $\text{Tr}_{\{2,3|c_{uk} \neq 0\}}$  stands for the summation over all eigenvalues of second and third stages, except for the  $uk$ -th ones corresponding to  $c_{uk} = 0$ . Also, the energy constraint (6) can be rewritten as

$$\text{Tr}_{\{2,3|c_{uk} \neq 0\}} \bar{V}_{\text{in}} = 2n \left[ N + \frac{1}{2} \right] - n_1 - \sum_k'' i_{uk}, \quad (27)$$

where  $i_{uk} = f_k^{-1}(x)$  and the double prime sum extends over  $uk$ -th eigenvalues of the second stage, such that  $c_{uk} = 0$ . Substituting Eq. (27) into Eq. (26) we get a transcendental equation for the single variable  $x$ . Since all unknown eigenvalues can be expressed through  $x$  (see Eqs. (25)) we can formally arrive at  $\underline{C}$ .

To the zeroth-order approximation (3) the relation  $i_{uk} = f_k^{-1}(x)$  (see Eq. (22)) gives

$$i_{uk} \approx i_{uk}^{(0)} = \frac{\sqrt{\eta^2 + 16(1-\eta)xe_{u,k}} - \eta}{8(1-\eta)e_{u,k}}, \quad (28)$$

which allows us to express  $\underline{C}$  as function of solution of only one algebraic equation for one variable  $x$ . By considering the term  $1/v^2$  of the decomposition (2) in Eq. (22), we obtain the first-order approximation for the relation  $i_{uk} = f_k^{-1}(x)$ . Since Eq. (22) cannot be exactly solved within this approximation, we solve it in neighborhood of the zeroth-order solution as linear perturbation. Thus, substituting  $i_{uk}^{(0)} + \varepsilon_{uk}$  in Eq. (22) instead of  $i_{uk}$  and solving for  $\varepsilon_{uk}$  we obtain the first-order approximation  $i_{uk}^{(1)} = i_{uk}^{(0)} + \varepsilon_{uk}$ , where

$$\begin{aligned} \varepsilon_{uk} = & (x - o_{uk})(x - o_{u,k})o_{uk}i_{uk}^{(0)} \\ & \times \{ [2(o_{uk}^2 + x^2 - (o_{uk} + o_{u,k})x) + (12o_{uk}^2 + 1)\nu_k^2]i_{uk}^{(0)}\eta \\ & - 2(1 + 12\nu_k^2)o_{uk}^2x \}^{-1} \end{aligned} \quad (29)$$

where all eigenvalues are calculated to zeroth-order approximation.

Finally, let us discuss on how to find the correct distribution of modes over the stages. As far as KKT-conditions themselves do not provide effective method to find stages distribution, we need to use some *a priori* properties to write an effective algorithm. Since we have conjectured (supported by numerics) that  $c_{uk}$ -eigenvalues are monotonic functions of  $N$ , the optimum for each mode has to be the third stage, or the second if the third is not admissible, or even the first if also the second one is not admissible. The third stage implies the existence of a solution inside some volume, the second stage a solution on the surface of that volume, while the first stage a solution on the edges of that surface. If the value of  $N$  is large enough, the solution is always inside a volume, so that all modes are in the third stage. By decreasing the value of  $N$ , the eigenvalues  $c_{uk}$  pass from the volume to the wall, and then to the edges. On the basis of these arguments we can develop two algorithms: *static* and *dynamic*.

The first step for both algorithms is the same: we check the positivity of all  $c_{uk}$  we have found through the relations (18). If all of them are positive,  $\underline{C}$  is given by the explicit analytical relation (20) and the problem is already solved. If this is not the case, we move on to the next steps.

- The static algorithm gives the following continuation on the bases of  $c_{uk}$ -signs found in the first step. We have to ascribe the third stage to modes with all

positive  $c_{uk}$ -eigenvalues, the second stage to modes with only one negative eigenvalue  $c_{uk}$  and the first stage to modes with both negative  $c_{uk}$ . All negative  $c_{uk}$  should be marked as already found and equal to zero. According to this distribution the set of KKT-equations should be formally solved (once again). If some second stage modes do not have a solution according to Eq. (22) they have to be marked as belonging to the first stage. Here we neglect the condition (24) for second stage modes, as violation of it implies negative  $c_{uk}$ , thus their stage will be changed to the first one by the algorithm itself. If all found  $c_{uk}$  are finally non-negative the problem is solved. If it is not, the procedure is iterated up until all  $c_{uk}$  will become non-negative. This algorithm can be then considered as a consecutive correction of stages distribution.

- The dynamic algorithm continues after the first step as follows. We solve the transcendental equation on  $x$  by choosing different stages distributions for different values of  $x$  (during iterations). In particular, each mode at beginning is calculated as belonging to the third stage for the current value of  $x$ . If this leads to negative  $c_{uk}$ , the mode is marked as belonging to the second or the first stage in analogy with the static algorithm (if some second stage modes violate Eq. (24) their stage should be marked as the first one). Thus, we make stages distribution admissible, in quantum sense, for every value of  $x$ . The distribution of stages corresponding to a root of the equation for  $x$  is considered to be valid and can be used to calculate the correct eigenvalues.

Both static and dynamic algorithms always yield the same stages distribution. The difference between the two algorithms can be clarified as follows. We have two effective unknown “variables” for KKT-equations: stages distribution and  $x$ . One of these variables has to be set as internal and the another one as external during maximization of  $\chi_n$ . The static algorithm uses  $x$  as internal variable, while the dynamic algorithm uses stages distribution for that. Since dynamic algorithm is usually faster, below we will make use of it. Also notice that the eigenvalues  $i_{uk}$  are always obtained as positive through these algorithms, therefore we did not specified their positivity in KKT-conditions.

If number of channel uses tends to infinity the discussed procedure can be properly generalized by changing the transcendental equations to equations on functions (spectral densities). However, if the considered model has some simple symmetries over stages, the general solution can be simplified by considering some parameters which mark the boundaries of stages. Below, we will show an example along this line.

The solution of KKT-conditions can be interpreted as “quantum water filling” in analogy with usual (classical) “water filling” introduced for classical Gaussian channels with memory (see e.g. [17]). The dependence of the found

spectral densities (also symplectic ones) from  $N$  is similar to filling a vessel with water. The form of the vessel is defined by the model  $V_{\text{env}}$  and transmissivity  $\eta$ . The symplectic spectral density  $\overline{\nu}_\xi$  goes always up by increasing  $N$  (with respect to  $\nu_\xi(N=0)$ ), while  $\nu_\xi$  goes always down (or does not change). For environment models showing correlation (memory) among modes, the presence of the second stage gives rise to capillary effects on the edges of the vessel resulting to a “water level” with meniscus form.

Below we apply the developed approach to channels whose environment is described by a covariance matrix

$$V_{\text{env}} = \left( N_{\text{env}} + \frac{1}{2} \right) \begin{bmatrix} e^{s\Omega} & 0 \\ 0 & e^{-s\Omega} \end{bmatrix}, \quad (30)$$

where  $N_{\text{env}}$  is the average number of *thermal* photons per mode in the environment,  $\Omega$  is a real symmetric  $n \times n$  matrix and  $s \in \mathbb{R}$  is a parameter describing the environment properties.

#### IV. THE MEMORYLESS CHANNEL

We first consider the case of  $\Omega = I$  describing a memoryless channel with a thermal and squeezed environment. The first stage for one channel use is always degenerate and corresponds to  $N = 0$  and  $C = 0$ . The case of the third stage, which includes values of  $N \gg s$ , results to an explicit formula for the lower bound on the classical capacity (see Eqs. (18)–(20)):

$$\underline{C} = g[\eta N + (1 - \eta)M_{\text{env}}] - g[(1 - \eta)N_{\text{env}}], \quad (31)$$

where, from the definition (21),

$$M_{\text{env}} = \left( N_{\text{env}} + \frac{1}{2} \right) \cosh s - \frac{1}{2}. \quad (32)$$

This analytical result generalizes that of [3, 5].

Considering the lower bound (31) as a function on the set of environment models with fixed  $M_{\text{env}}$ , we see that it shows a *symmetry breaking over quadratures*. In fact despite the symmetry of all equations over quadratures, the maximum of lower bound is achieved when  $e_q \neq e_p$ .

The solution for the second stage is  $c_u = 0$ ,  $c_{u_*} = 2N + 1 - i_u - 1/(4i_u)$ ,  $i_{u_*} = 1/(4i_u)$ , where  $i_u$  is a root of the transcendent equation

$$\begin{aligned} & \left( \frac{1}{o_u} - \frac{1}{a_{u_*}} \right) \overline{\nu} \frac{\partial g(\overline{\nu} - 1/2)}{\partial \overline{\nu}} \\ &= \left( \frac{1}{o_u} - \frac{i_{u_*}}{i_u o_{u_*}} \right) \nu \frac{\partial g(\nu - 1/2)}{\partial \nu}. \end{aligned} \quad (33)$$

The condition  $c_{u_*} > 0$  restrict the admissible region for  $i_u$  to the interval

$$N + \frac{1}{2} - \sqrt{N^2 + N} < i_u < N + \frac{1}{2} + \sqrt{N^2 + N}. \quad (34)$$

To zeroth-order approximation (3) the solution of Eq. (33) is

$$i_u \approx i_u^{(0)} = \frac{1}{2} \left[ \sqrt{1 + (2N + 1)\phi + \phi^2/4} - \phi/2 \right], \quad (35)$$

where  $\phi = e_{u_*}^{-1}\eta/(1 - \eta)$ . Hence, the classical capacity with this approximation is expressible in an explicit form.

By considering the term  $1/v^2$  of decomposition (2) in Eq. (33), we obtain the equation for the first-order approximation. Proceeding in analogy with the general case (see Sec. III) we find the first-order approximation  $i_u^{(1)} = i_u^{(0)} + \varepsilon_u$ , where

$$\varepsilon_u = \frac{\eta a_{u_*} o_u i_u^{(0)} c_{u_*} (a_{u_*} - o_u)}{2[\eta^2(o_u^2 + a_{u_*}^2 - a_{u_*} o_u) i_u^{(0)} c_{u_*} - a_{u_*}^2 o_u^2 (12\nu^2 + 1)]} \quad (36)$$

and all eigenvalues in Eq. (36) are calculated through zeroth-order approximation. Notice that the first-order approximation considered in Sec. III does not coincide with  $i_u^{(1)}$  found. Actually, we have to jointly solve two equations in the case of general method applied to one channel use: the transcendent equation (22) and the equation for  $x$  (see Eqs. (26) and (27)), where only Eq. (22) is approximated. However, the equation for  $x$  becomes analytically solvable in the case of one channel use, allowing us to reduce the original two equations to a single one given by (33).

The formal algorithm to find the lower bound on the capacity is similar to general case considered in Sec. III. At first we solve KKT-conditions by considering the mode to be in the third stage. If one of the found  $c_u$ -eigenvalues is negative we mark it as already found and set equal to zero. Then, we solve again the KKT-conditions by considering the mode to be in the second stage. This always brings us to the correct eigenvalues allowing us to find the lower bound on the capacity.

Writing the input covariance matrix in the same form of (30) (with  $\Omega = I$ ), by the replacements  $N_{\text{env}} \rightarrow N_{\text{in}}$  and  $s \rightarrow r$ , we can relate the optimal degree of input squeezing  $r_{\text{opt}}$  to the degree of environment squeezing  $s$ . It follows from Eqs. (19) and (35) that  $r_{\text{opt}} = s$  for the third stage and

$$r_{\text{opt}} \approx r_{\text{opt}}^{(0)} = \text{sign}(s) \ln \left[ \sqrt{1 + (2N + 1)\phi + \phi^2/4} - \phi/2 \right]$$

for the second stage in the zeroth-order approximation. At the transition point between stages there is a *kink* in the function  $r_{\text{opt}}(s)$  (see Fig.1-right). It reflects the fact that different stages correspond to solution of different systems of equations.

The lower bound  $\underline{C}$  found by the exact and approximate analytical solutions is shown in Fig.1-left as function of  $s$  and for different values of  $\eta$ . One can see that in the limit of large  $s$  the lower bound on the capacity only depends on the energy constraint  $N$ , specifically (see also [16])

$$\lim_{s \rightarrow \infty} \underline{C}(s, \eta, N, N_{\text{env}}) = \log_2(2N + 1), \quad (37)$$

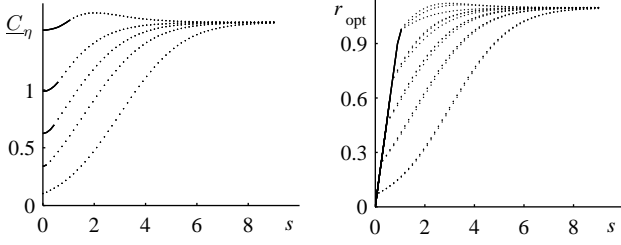


FIG. 1: Lower bound on the classical capacity  $\underline{C}$  (left) and optimal input squeezing  $r_{\text{opt}}$  (right) vs  $s$ , for values of  $\eta$  going from 0.1 (bottom curve) to 0.9 (top curve) with step 0.2. The values of the other parameters are  $N = N_{\text{env}} = 1$ . On the left, exact and approximate solutions almost coincide. On the right, for each value of  $\eta$ , the zeroth-order approximation, the first-order approximation and the exact solution are plotted (bottom to top). Solid and dotted parts of the curves correspond to the third and second stages, respectively.

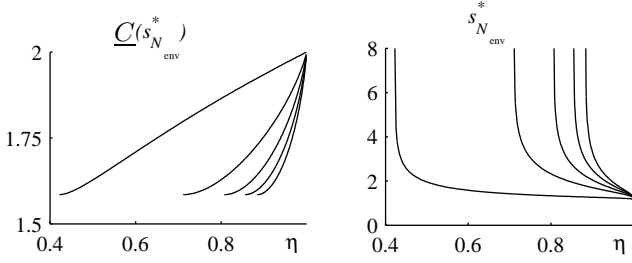


FIG. 2: Lower bound on the classical capacity  $\underline{C}(s_{N_{\text{env}}}^*)$  (left) and optimal environment squeezing  $s_{N_{\text{env}}}^*$  (right) vs  $\eta$  for values of  $N_{\text{env}}$  going from 0 (left curve) to 2 (right curve) with step 0.5. The value of the other parameter is  $N = 1$ .

explaining why all curves  $\underline{C}_\eta(s)$  flow together to the same value when  $s \rightarrow \infty$  (see Fig.1-left). The optimal degree of squeezing in the environment  $s^*$  (which maximizes the lower bound on the capacity) is shown in Fig.2-right and Fig.4-right as function of  $\eta$  for different values of  $N_{\text{env}}$  and  $N$  respectively. The lower bound on the capacity corresponding to these values of  $s^*$  is plotted in Fig.2-left and Fig.4-left. The *critical transmissivity*  $\eta^*$ , representing a threshold between transmissivities with finite and infinite values of  $s^*$ , is drawn in Fig.3-right as function of  $N_{\text{env}}$  for different values of  $N$ .

The points  $(a_q, a_p)$  and  $(o_q, o_p)$  are plotted in Fig.3-left versus  $N$  for fixed values of  $s, \eta$  and  $N_{\text{env}}$ . The resulting curves show the geometry of the stage transitions and visualize the “quantum water filling” effect for one channel use.

## V. A CHANNEL WITH MEMORY

Let us now consider an environment model (30) with

$$\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}; \quad i, j = 1, \dots, n \quad (38)$$

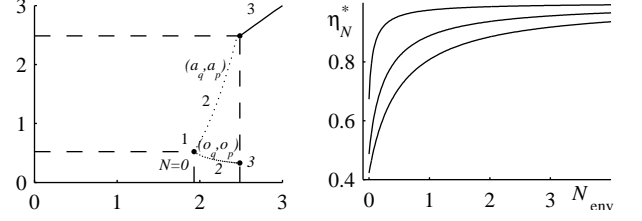


FIG. 3: On the left, quantities  $(o_q, o_p)$  and  $(a_q, a_p)$  vs  $N$ . The values of other parameters are  $N_{\text{env}} = s = 1$ ,  $\eta = 0.6$ . The second stage is marked by dotted curves. The first stage is a single point at  $N = 0$ . The whole third stage for the quantity  $(o_q, o_p)$  is mapped into a single point, as  $V_{\text{in}}$  does not depend on  $N$  in the third stage. On the right, critical transmissivity  $\eta^*$  vs  $N_{\text{env}}$  for values of  $N = 1, 500, 1000$  from bottom to top.

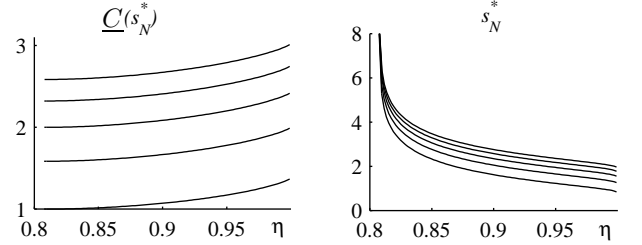


FIG. 4: Lower bound on the classical capacity  $\underline{C}(s_N^*)$  (left) and squeezing parameter  $s_N^*$  (right) vs  $\eta$  for values of  $N$  going from 0.5 (bottom curve) to 2.5 (top curve) with step 0.5. The value of the other parameter is  $N_{\text{env}} = 1$ .

describing a lossy bosonic channel with memory [11]. Here  $s$  represents the degree of correlation among environment modes. We are interested in the asymptotic behavior of this channel. That implies to take the limit  $n \rightarrow \infty$  in the equations of Sec. III. It can be treated for some relations as the limit of Riemann sums resulting to the integral expressions. Instead of a set of equations on eigenvalues we get a set of equations on functions which are spectral densities for the involved (infinite-dimensional) matrices. Below we denote the spectral densities by the same symbols as proper eigenvalues, but written in calligraphic and replacing the mode number  $h$  by a continuous parameter  $\xi$ , i.e.,  $i_{uh} \rightarrow \mathcal{I}_{u\xi}$ ,  $o_{qh} \rightarrow \mathcal{O}_{q\xi}$ , etc.

Suppose that all modes belong to the third stage, which holds true if (see Eq. (18))

$$w := \frac{1}{2|s|} \ln \frac{\eta(2N+1) + (1-\eta)(2N_{\text{env}}+1)I_0(2s)}{\eta + (1-\eta)(2N_{\text{env}}+1)} \geq 1, \quad (39)$$

where  $I_0$  is the modified Bessel function of the first kind and zero-order. The lower bound  $\underline{C}$  in this case is given by Eq. (31) after a formal replacement  $\cosh s \rightarrow I_0(2s)$  in Eq. (32). This example explicitly shows the possibility of an enhancement of the lower bound on the classical capacity with increasing degree of memory  $s$ .

It is convenient to use the parameter  $\xi$  as arising from

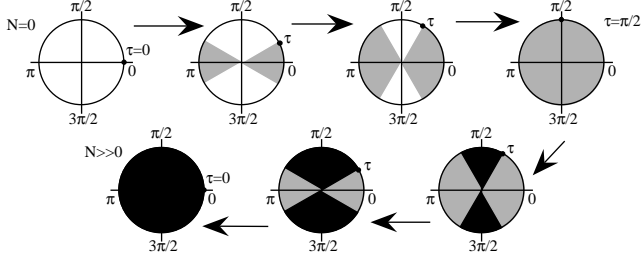


FIG. 5: Schematic representation of the “quantum water filling” for the model  $\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$ . The angle  $\xi$  parametrizing the spectral density corresponds to polar angle. White, grey and black sectors correspond to the first, second and third stages, respectively. Arrows show change of stages with increasing of  $N$ . The parameter  $\tau$  marks the points of stages transition.

the spectrum of  $V_{\text{env}}$ -matrix [11]

$$\mathcal{E}_{u\xi} = \left( N_{\text{env}} + \frac{1}{2} \right) e^{\pm 2s \cos \xi}, \quad (40)$$

labeling both modes (if  $\xi \in [0, \pi]$ ) and eigenvalues (if  $\xi \in [0, 2\pi]$ ). Plus and minus in Eq. (40) stand for  $u = q$  and  $u = p$ , respectively. Due to the mirror symmetry of eigenvalues (40) over quadratures, the symplectic spectrum and the stages distribution have to be symmetric with respect to the point  $\pi/2$ , therefore we restrict ourselves to consider spectral densities only defined in the interval  $[0, \pi/2]$ .

If  $w < 1$  we can have one of the following stages distributions according to the results of numerical calculations for finite  $n$  (accordingly to the procedure of Sec. III):

- i) a mixture of second and third stages  $(2, 3, 2)$ ;
- ii) a mixture of second and first stages  $(2, 1, 2)$ ;
- iii) all modes belonging to the second stage  $(2, 2, 2)$  which happens for a single value  $N_2$  of the parameter  $N$ , given  $s$ ,  $\eta$  and  $N_{\text{env}}$ .

If  $N > N_2$  or  $N < N_2$  we have the  $(2, 3, 2)$  or  $(2, 1, 2)$  case with the center of the interval  $[0, \pi]$  filled by the third or the first stage, respectively. We label the point of stages transition by  $\tau \in [0, \pi/2]$ . The possible stages distributions and dependence of  $\tau$  from  $N$  are sketched in Fig.5. The continuity of the spectral density  $\mathcal{C}_{u\xi}$  at points of stages transition  $\tau$  requires to hold  $\mathcal{A}_{u\tau} = \mathcal{O}_{u\tau}$  which can be rewritten as

$$x = x(\tau) = \eta \mathcal{I}_{u\tau} + (1 - \eta) \mathcal{E}_{u\tau}. \quad (41)$$

Here  $u = q$  gives  $\mathcal{I}_{q\tau} = e^{2s \cos \tau} / 2$  (see Eq. (18)) for  $(2, 3, 2)$  and  $u = p$  gives  $\mathcal{I}_{p\tau} = 1/2$  for  $(2, 1, 2)$  (we use different quadratures in these cases because of either  $\mathcal{C}_{q\xi}$  or  $\mathcal{C}_{p\xi}$  has a transition point belonging to the interval of  $\tau$ ).

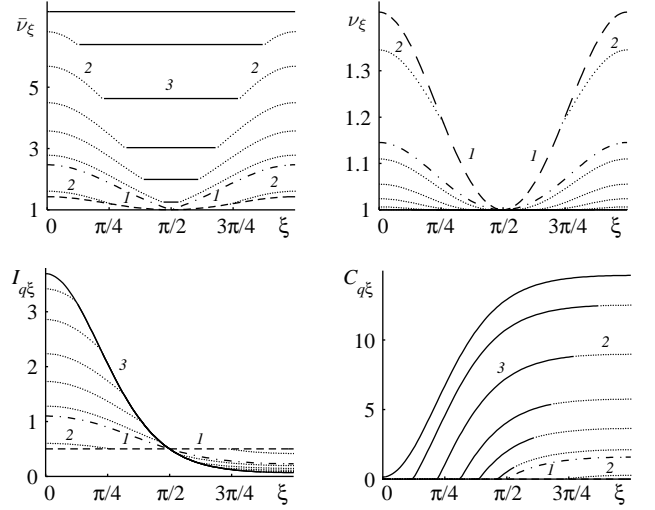


FIG. 6: Going from top-left clockwise the spectral densities  $\bar{\nu}_\xi, \nu_\xi, \mathcal{C}_{q\xi}, \mathcal{I}_{q\xi}$  (for  $\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$ ) are plotted vs the parameter  $\xi$  for  $N = 0, 0.05, 0.67, 1, 2, 3.5, 6, 9, 11$  (from bottom to top curve for quantities  $\bar{\nu}_\xi, \mathcal{C}_{q\xi}, \mathcal{I}_{q\xi}$ , and from top to bottom curve for quantity  $\nu_\xi$ ). Solid, dotted and dashed parts of curves correspond to third, second and first stages, respectively. Dash-dotted curve corresponds to the case of all modes belonging to the second stage. The values of other parameters used are  $N_{\text{env}} = s = 1, \eta = 0.5$ .

Then, the transcendent equation for  $x$  (see Eqs. (26) and (27)) can be rewritten as an equation for  $\tau$

$$\eta \left[ N + \frac{\tau_1}{\pi} - \frac{1}{\pi} \int_0^\tau \mathcal{I}_{q\xi} d\xi \right] + \frac{1 - \eta}{\pi} \int_0^{\tau_2} \mathcal{E}_{p\xi} d\xi = \frac{\tau_2}{\pi} x, \quad (42)$$

where  $(\tau_1, \tau_2)$  is equal to  $(\tau, \tau)$  for  $(2, 1, 2)$  and to  $(\pi/2, \pi - \tau)$  for  $(2, 3, 2)$ . Moreover,  $x$  is given by Eq. (41) and  $\mathcal{I}_{q\xi}$  is the spectral density for the second stage which can be found as solution of functional equation obtained from Eq. (22) or its approximations (see Eqs. (28), (29)) after the replacements discussed at the beginning of this section. By substituting  $\tau = \pi/2$  in Eq. (42) we find  $N_2$ . Comparing it with the actual energy restriction we get correct stages distribution. Then, solving Eq. (42) with the found stages distribution we arrive at  $\tau$  and  $x$ . Finally,  $\underline{\mathcal{C}}$  is expressed through these parameters as follows (see Eqs. (7) and (9)):

$$\underline{\mathcal{C}} = \left( 1 - \frac{2}{\pi} \tau_3 \right) \left[ g \left( x - \frac{1}{2} \right) - g((1 - \eta) N_{\text{env}}) \right] + \frac{2}{\pi} \int_0^\tau \left[ g \left( \sqrt{x \mathcal{O}_{q\xi}} - \frac{1}{2} \right) - g \left( \sqrt{\mathcal{O}_{q\xi} \mathcal{O}_{p\xi}} - \frac{1}{2} \right) \right] d\xi,$$

where

$$\mathcal{O}_{q\xi} = \eta \mathcal{I}_{q\xi} + (1 - \eta) \mathcal{E}_{q\xi}, \quad (43)$$

$$\mathcal{O}_{p\xi} = \frac{\eta}{4} \mathcal{I}_{q\xi}^{-1} + (1 - \eta) \mathcal{E}_{p\xi}, \quad (44)$$

$\tau_3$  is equal to  $\pi/2$  for  $(2, 1, 2)$  and to  $\tau$  for  $(2, 3, 2)$ .

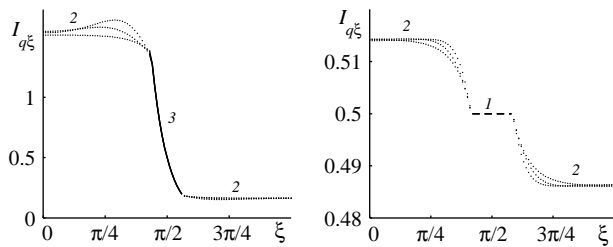


FIG. 7: Exact solution, first-order and zeroth-order approximations for spectral density  $\mathcal{I}_{q\xi}$  vs  $\xi$  for  $N = 1$  (left) and  $N = 0.01$  (right). The values of other parameters are  $N_{\text{env}} = 0.5$ ,  $s = 2.5$ ,  $\eta = 0.95$ . Solid, dotted and dashed parts of curves correspond to third, second and first stages, respectively. Functions with maximum and minimum variations correspond to exact solution and zeroth-order approximation, respectively.

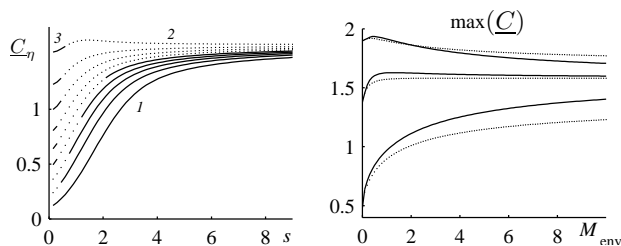


FIG. 8: On the left, the quantity  $\underline{C}$  is plotted vs  $s$  for values of  $\eta$  going from 0.1 (bottom curve) to 0.9 (top curve) with step 0.1. The values of the other parameters are  $N = N_{\text{env}} = 1$ . Solid parts of curves correspond to the third and first stages, respectively. Dotted part of curves correspond to the second stage. On the right, the maximum of  $\underline{C}$  over  $V_{\text{env}}$  is plotted vs  $M_{\text{env}}$  for values of  $\eta = 0.1, 0.5, 0.9$  going from bottom to top curve. Solid and dotted curves corresponds to  $\Omega = I$  and  $\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$ , respectively. The value of the other parameter is  $N = 1$ .

The “quantum water filling” effect for the considered model is shown in Fig.6 for symplectic spectral densities  $\bar{\nu}_\xi$ ,  $\nu_\xi$  and spectral densities  $\mathcal{C}_{q\xi}$ ,  $\mathcal{I}_{q\xi}$ . Graphs of  $\mathcal{I}_{q\xi}$  calculated through exact solution, zeroth-order and first-order approximations are shown in Fig.7. Despite some visible difference between exact and approximate spectral densities the corresponding symplectic spectral densities are almost equal, thus resulting to the difference less than 0.05% between  $\underline{C}$  calculated exactly and approximately. The small value of this difference comes from the fact that the Holevo- $\chi$  has zero derivative over eigenvalues of  $V_{\text{in}}$  and  $V_{\text{cl}}$  in the neighborhood of solutions of KKT-conditions (as they are equations for optimiza-

tion problem).

Finally, let us discuss the role of squeezing and memory in lossy bosonic channel. Suppose, that the average amount of photons in the environment  $M_{\text{env}}$  is fixed and we want to compare the quantity  $\underline{C}$  for  $\Omega \neq I$  with that of  $\Omega = I$ . As far as the Holevo- $\chi$  is symmetric over modes one can expect that the lower bound on one shot capacity for  $\Omega = I$  will always be higher. However, this is not true as results from the *symmetry breaking over modes*. Actually, we can see this from Fig.8-right where  $\underline{C}$  maximized over parameters  $s, N_{\text{env}}$  for memory and for memoryless cases is plotted versus  $M_{\text{env}}$ . In Fig.8-left the lower bound on the classical capacity for memory model is plotted versus  $s$  for different values of  $\eta$ . The limit of the one shot capacity (37) when  $s \rightarrow \infty$  is still valid. Presumably, as we can see from Fig.8-right, this limit is valid in the more general case of function  $\max_{V_{\text{env}}}(\underline{C})$  when  $M_{\text{env}} \rightarrow \infty$ .

## VI. CONCLUSION

Summarizing, we have developed a powerful and versatile tool for the estimation of Gaussian quantum channels capacities. It is based on the perturbative expansion of the von Neumann entropy of Gaussian states as function of the symplectic eigenvalues of the quadratures covariance matrix. We have applied this method to the classical capacity of a lossy bosonic channel. The found lower bounds are reliable rates of communication which can be achieved using solely Gaussian states for encoding classical information. By providing upper bounds it was shown that, in some settings, Gaussian state encoding is indeed optimal. Thus we have generalized the results of [5] and partially those of [8] and [16]. The expansion of the von Neumann entropy can be as well applied for the evaluation and optimization of other entropic function. Of a special interest is the case of the quantum capacity. In particular, it has been proven [7] that Gaussian state encoding is optimal for reaching the quantum capacity of the so-called *degradable* Gaussian channels.

## Acknowledgment

The research leading to these results has received funding from the European Commission’s seventh Framework Programme (FP7/2007-2013) under grant agreement no. 213681. O. P. thanks V. G. Zborovskii for fruitful discussions.

[1] Holevo A. S., “On the mathematical theory of quantum communication channels”, *Probl. Inf. Transm.* vol. 8, pp. 62-71, 1972.

[2] Bennett C. H., and Shor P. W. “Quantum information theory”, *IEEE Trans. Inf. Th.* vol. 44, pp. 2724-2742, 1998.



- [3] Holevo A. S., and Werner R. F., “Evaluating capacities of Bosonic Gaussian channels”, *Phys. Rev. A* vol. 63, pp. 032312-1–032312-14, 2001.
- [4] Braunstein S. L., and Pati A. K., *Quantum Information Theory with Continuous Variables*, Dordrecht, Kluwer Academic, 2003.
- [5] Giovannetti V., Guha S., Lloyd S., Maccone L., Shapiro J. H., and Yuen H. P., “Classical capacity of the lossy bosonic channel: the exact solution”, *Phys. Rev. Lett.* vol. 92, pp. 027902-1–027902-4, 2004.
- [6] Giovannetti V., Lloyd S., Maccone L., and Shor P. W., “Entanglement assisted capacity of the broadband lossy channel”, *Phys. Rev. Lett.* vol. 91, pp. 047901-1–047901-4, 2003.
- [7] Wolf M. M., Perez-Garcia D., and Giedke G., “Quantum Capacities of Bosonic Channels”, *Phys. Rev. Lett.* vol. 98, pp. 130501-1–130501-4, 2007.
- [8] Lloyd S., Giovannetti V., Maccone L., Cerf N. J., Guha S., Garcia-Patron R., Mitter S., Pirandola S., Ruskai M. B., Shapiro J. H., and Yuan H., “Proof of the bosonic minimum output entropy conjecture”, *arxiv:0906.2758[quant-ph]*, 2009.
- [9] Giovannetti V., and Mancini S., “Bosonic Memory Channels”, *Phys. Rev. A* vol. 71, pp. 062304-1–062304-6, 2005.
- [10] Holevo A. S., Shohma M., and Hirota O., “Capacity of quantum Gaussian channels”, *Phys. Rev. A* vol. 59, pp. 1820-1828, 1999.
- [11] Pilyavets O. V., Zborovskii V. G., and Mancini S., “A Lossy Bosonic Quantum Channel with Non-Markovian Memory”, *Phys. Rev. A* vol. 77, pp. 052324-1–052324-8, 2008.
- [12] Holevo A. S., “Quantum coding theorems”, *Russ. Math. Surveys*, vol. 53, pp. 1295-1331, 1998.
- [13] Schumacher B., and Westmoreland M. D., “Sending classical information via noisy quantum channels”, *Phys. Rev. A* vol. 56, pp. 131-138, 1997.
- [14] Bowen G., Mancini S., “Quantum channels with a finite memory”, *Phys. Rev. A* vol. 69, pp. 012306-1–012306-6, 2004.
- [15] Kretschmann D., Werner R. F., “Quantum channels with memory”, *Phys. Rev. A* vol. 72, pp. 062323-1–052324-19, 2005.
- [16] Lupo C., Pilyavets O. V., and Mancini S., “Capacities of lossy bosonic channel with correlated noise”, *New J. Phys.* vol. 11, pp. 063023-1–063023-18, 2009.
- [17] Cover T. M., and Thomas J. A., *Elements of Information Theory*, New York, Wiley-Interscience, 1991.